3 Rindler Space and Hawking Radiation

The next couple of lectures are on Hawking radiation. There are many good references to learn this subject, for example: Carroll's GR book Chapter 9; Townsend gr-qc/9707012; Jacobson gr-qc/0308048. Therefore in these notes I will go quickly through some of the standard material, but slower through the material that is hard to find elsewhere. I strongly recommend reading Carroll's chapter too.

Hawking radiation is a feature of QFT in curved spacetime. It does not require that we quantize gravity – it just requires that we quantize the perturbative fields on the black hole background. In fact we can see very similar physics in flat spacetime.

3.1 Rindler space

2d Rindler space is a patch of Minkowski space. In 2D, the metric is

$$ds^2 = dR^2 - R^2 d\eta^2 \ . \tag{3.1}$$

There is a horizon at $R = 0$ so these coordinates are good for $R > 0, \eta =$anything.

Notice the similarity to polar coordinates on R^2, if we take $\eta \to i\phi$. This suggests the following coordinate change from 'polar-like' coordinates to 'Cartesian-like' coordinates,

$$x = R\cosh\eta, \quad t = R\sinh\eta \ . \tag{3.2}$$

The new metric is just Minkowski space $R^{1,1}$,

$$ds^2 = -dt^2 + dx^2 \ . \tag{3.3}$$

Looking at (3.2), we see $x^2 - t^2 = R^2 > 0$, so the Rindler coordinates only cover the patch of Minkowski space with

$$x > 0, \quad |t| < x \ . \tag{3.4}$$

This is the 'right wedge', which covers one quarter of the Penrose diagram.

Higher dimensional Rindler space is

$$ds^2 = dR^2 - R^2 d\eta^2 + d\vec{y}^{\,2} \ , \tag{3.5}$$

so we can map this to a patch of $R^{1,D-1}$ by the same coordinate change. The other coordinates just come along for the ride.

A 'Rindler observer' is an observer sitting at fixed R. This is not a geodesic — it is a uniformly accelerating trajectory. You can check this by mapping back to Minkowski space. Rindler observers are effectively 'confined' to a piece of Minkowski space, and they see a horizon at $R = 0$. This horizon is in many ways very similar to a black hole horizon.

Exercise: Rindler time translations are Minkowski boosts
Difficulty level: a few lines
$\zeta_{(\eta)} = \partial_\eta$ is an obvious Killing vector of Rindler space, since the metric is independent of η. By explicitly transforming this vector to Minkowski coordinates, show that it is a Lorentz boost.[15]

3.2 Near the black hole horizon

Black holes have an approximate Rindler region near the horizon. For example, start with the Schwarzschild solution

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_2^2, \qquad f(r) = 1 - \frac{2M}{r} \ . \tag{3.6}$$

Make the coordinate change

$$r = 2M(1+\epsilon^2), \quad \text{so} \quad f(r) \approx \epsilon^2 \tag{3.7}$$

[15]Reminder: The notation $\zeta = \partial_\eta$ means, in components, $\zeta^\mu \partial_\mu = \partial_\eta$, *i.e.*, $\zeta = \hat{\eta}$.

and expand the metric at small ϵ,

$$ds^2 = -\epsilon^2 dt^2 + 16M^2 d\epsilon^2 + 4M^2 d\Omega_2^2 + \cdots \tag{3.8}$$

The (t, ϵ) piece of this metric is Rindler space (we can rescale t and ϵ to make it look exactly like (3.1)).

Although 2d Rindler is a solution of the Einstein equations, the metric written in (3.8) (excluding the dots) is $R^{1,1} \times S^2$. This is not a solution of the Einstein equations. It is only an approximate solution for small ϵ.

3.3 Periodicity trick for Hawking Temperature

Now we will give a 'trick' to derive the Hawking temperature. The trick is to argue that that, in the black hole metric, the time coordinate must be periodic in the imaginary direction, and this imaginary periodicity implies that the black hole has a temperature. Actually, this trick is completely correct, and we will justify it later from the path integral, but don't expect this subsection to be very convincing yet!

First we want to argue that QFT at finite temperature is periodic in imaginary time, with periodicity

$$t \sim t + i\beta, \qquad \beta = 1/T \ . \tag{3.9}$$

We will return this in detail later, but for now one way to see it is by looking at the thermal Green's function[16]

$$G_\beta(\tau, x) \equiv -\ \mathrm{Tr}\ \rho_{thermal} T_E \left[O(\tau, x) O(0,0)\right]\ = -\frac{1}{Z}\ \mathrm{Tr}\ e^{-\beta H} T_E \left[O(\tau, x) O(0,0)\right], \tag{3.10}$$

where $\tau = it$ is Euclidean time, and T_E means Euclidean-time ordering (*i.e.,* put the

[16]This definition holds for $-\beta < \tau < \beta$. A good reference for the many types of thermal Green's functions is Fetter and Walecka, *Quantum Theory of Many-particle Systems*, 1971, in particular chapters 7 and 9. The function G_β defined here is equal to the Euclidean Green's function on a cylinder that we will discuss later (up to normalization).

larger value of τ on the left). This is periodic in imaginary time,[17]

$$\begin{aligned} G_\beta(\tau, x) &= -\frac{1}{Z}\ \mathrm{Tr}\ e^{-\beta H} O(\tau, x) O(0,0) && (3.11) \\ &= -\frac{1}{Z}\ \mathrm{Tr}\ O(0,0) e^{-\beta H} O(\tau, x) && (3.12) \\ &= -\frac{1}{Z}\ \mathrm{Tr}\ e^{-\beta H} O(\beta, 0) O(\tau, x) && (3.13) \\ &= G_\beta(\tau - \beta, x) && (3.14) \end{aligned}$$

Now returning to black holes, Rindler space (3.1) is related to polar coordinates, $dR^2 + R^2 d\phi^2$ with $\eta = i\phi$. Polar coordinates on R^2 are singular at the origin unless ϕ is a periodic variable, $\phi \sim \phi + 2\pi$. Therefore η is periodic in the imaginary direction,

$$\eta \sim \eta + 2\pi i \ . \tag{3.15}$$

Going back through all the coordinate transformations relating the near horizon black hole to Rindler space, this implies that the Schwarzschild coordinate t has an imaginary periodicity

$$t \sim t + i\beta\ , \qquad \beta \equiv 8\pi M\ . \tag{3.16}$$

Comparing to finite temperature QFT,

$$T = \frac{1}{\beta} = \frac{1}{8\pi M}\ . \tag{3.17}$$

This agrees with the Hawking temperature derived from the first law, (2.19) (setting $Q = 0$). As mentioned above, this derivation probably is not very convincing yet, but it is often the easiest way to calculate T given a black hole metric.

Exercise: Schwarzschild periodicity

Difficulty level: easy

Work through the coordinate transformations in section (3.2) to relate the Schwarzschild near-horizon to Rindler space, and show that (3.15) implies (3.16).

[17] 1st line: definition (we assume $0 < \tau < \beta$, so this is τ-ordered). 2nd line: cyclicity of trace. 3rd line: definition of time translation, $O(\tau, x) = e^{\tau H} O(0, x) e^{-\tau H}$. 4th line: definition.

Exercise: Kerr periodicity

Difficulty level: moderate – a few pages

Caveat: I have not checked whether my sign conventions for Ω in this problem agree with the conventions above in the 1st law, so if you get the opposite sign don't worry and please let me know.

Field theory at finite temperature and and angular potential is periodic in imaginary time, but with an extra shift in the angular direction:

$$(t, \phi) \sim (t + i\beta, \phi + i\beta\Omega) \ . \tag{3.18}$$

(a) Derive (3.18) by an argument similar to (3.11). The density matrix for QFT at finite temperature and angular potential is $\rho = e^{-\beta(H-\Omega J)}$ where J is the angular momentum.

(b) Starting from the Kerr metric (2.24), we will follow steps similar to section 3.2 to relate the near horizon to Rindler space. This will be easiest if you write $\Delta(r) = (r - r_+)(r - r_-)$ and work in terms of $r_\pm$ instead of plugging in all the M's and a's. Also, you can safely ignore that θ-direction by setting $\theta = \pi/2$ (you should come back at the end of the problem and convince yourself that this was reasonable).

Plug in $r = r_+(1 + \epsilon^2)$ and expanding in ϵ. You should find something of the form

$$a d\epsilon^2 + (b d\phi - c dt)^2 + \epsilon^2(e d\phi + f dt + \cdots) + \cdots \ . \tag{3.19}$$

where a, b, c, e, f are some constants.

(c) To find the correct periodicity in this situation, define the corotating angular coordinate $\tilde{\phi} = b\phi - ct$ (this coordinate rotates with the horizon). Now the usual Rindler argument implies that t has an imaginary identification with $\tilde{\phi}$ held fixed. Translate this identification back into t, ϕ coordinates and compare to (3.18) to read off β and Ω. Check that your answers agree with the ones you derived from the first law in the exercise around (2.24).

3.4 Unruh radiation

Suppose our spacetime is Minkowski space, in the vacuum state. An observer on a worldline of fixed x will not observe any excitations. So, if the observer carries a thermometer, then the thermometer will read 'temperature = 0' for all time. More generally, if the observer carries any device with internal energy levels that can be excited by interacting with whatever matter fields exist in the theory (an *Unruh detector*), then this device will forever remain in its ground state.

However, now consider a uniformly accelerating observer with acceleration a. This corresponds to a Rindler observer sitting at fixed $R = 1/a$. We will show that in the same quantum state – the Minkowski vacuum – this observer feels a heat bath at temperature $T = \frac{a}{2\pi}$.

No unique vacuum

Thus the Minkowski vacuum is not the same as the Rindler vacuum. This is a general feature of quantum field theory in curved space (although in this example spacetime is flat!). In general, there is no such thing as *the* vacuum state, only the vacuum state according to some particular observer. The reason for this is the following. In QFT, we expand quantum fields in energy modes,

$$\hat{\phi} = \sum_{\omega>0,k} \left(a_{\omega,k} e^{i\omega t - ikx} + a^\dagger_{\omega,k} e^{-i\omega t + ikx} \right) \ . \tag{3.20}$$

The 'vacuum' is defined as the state annihilated by the negative energy modes:

$$a_\omega |0\rangle = 0 \ , \tag{3.21}$$

and excitations are created by the positive-energy modes $a^\dagger$.

The ambiguity comes from the fact that energy is observer dependent. The energy is the expectation value of the Hamiltonian; and the Hamiltonian is the operator that generates time evolution

$$\frac{i}{\hbar}[H, O] = \partial_t O \ . \tag{3.22}$$

Therefore the Hamiltonian depends on a choice of time t. In GR, we are free to call any timelike direction t. Different choices of this coordinate correspond to different choices of Hamiltonian, and therefore different notions of positive energy, and therefore different notions of vacuum state.

(FIGURE: Time slicings of Minkowski vs Rindler)

The Unruh temperature

The quick way to the Unruh temperature is the imaginary-time periodicity trick. Since the Rindler time coordinate η is periodic under $\eta \sim \eta + 2\pi i$, this looks like a temperature $T = \frac{1}{2\pi}$. This is the temperature associated to the time translation ∂_η. In other words, if H_η is the Hamiltonian that generates η-translations, then the periodicity trick tells us that the density matrix for fields in Rindler space is

$$\rho_{Rindler} = e^{-2\pi H_\eta} \ . \tag{3.23}$$

The proper time of a Rindler observer at fixed $R = R_0$ is

$$d\tau = R_0 d\eta \ . \tag{3.24}$$

Therefore R_0 is the redshift factor, and the temperature actually observed (say by a thermometer) is

$$T_{thermometer} = \frac{1}{\sqrt{g_{\eta\eta}}}\frac{1}{2\pi} = \frac{1}{2\pi R_0} = \frac{a}{2\pi} \ . \tag{3.25}$$

Where in this argument did we actually decide which state we are in? There are excitations of Rindler/Minkowski space – say, a herd of elephants running by – where an observer will certainly not measure a uniform heat bath of thermal radiation! The answer is that by applying the periodicity trick, we have actually selected one particular very special state, (3.23). What we still need to show is that this state is in fact the Minkowski vacuum.

First, some FAQ:[18]

- *Does a Minkowski observer see that the Rindler observer is detecting particles?* Yes, the Minkowski observer can see that the Rindler observer's particle detector

[18] See http://www.scholarpedia.org/article/Unruh_effect for further discussion and references.

is clicking. *How does that make any sense if the Minkowski observer doesn't see the particles being absorbed?* The Minkowski observer actually sees the Rindler observer's detector *emit* a particle when it clicks. This can be worked out in detail, but the easy way to see this is that the absorption of a Rindler mode changes the quantum state of the fields; the only way to change the vacuum state is to excite something.

- *Doesn't this violate conservation of energy?* No, the Rindler observer is uniformly accelerating. So this observer must be carrying a rocket booster. From the point of view of a Minkowski observer, the rocket is providing the energy that excites the Rindler observer's thermometer, and causes Minkowski-particle emission.

- *How hot is Unruh radiation?* Not very hot. In Kelvin, $T = \hbar a/(2\pi c k_B)$. To reach $1K$, you need to accelerate at $2.5 \times 10^{20} m/s^2$.

- *According to the equivalence principle, sitting still in a uniform gravitational field is the same as accelerating. So do observers sitting near a massive object see Unruh radiation?* Only if it's a black hole (as discussed below). Observers sitting on Earth do not see Unruh radiation because the quantum fields near the Earth are not in the state (3.23). This is similar to the answer to the question 'why doesn't an electron sitting on the Earth's surface radiate?'.

Back to Minkowski

One very explicit method to show that the thermal Rindler state is in fact the Minkowski vacuum is to compare the Rindler modes to the Minkowski modes, and check that imposing $a_w^{Minkowski}|0\rangle = 0$ leads to the state (3.23) in the Rindler half-space. This is already nicely written in Carroll's GR book and many other places so I will not bother writing it here. Read the calculation there. But don't be fooled – that method gives the impression that the Rindler temperature has something to do with the modes of a free field in Minkowski space. This is not the case. The Rindler temperature is fixed by symmetries, and holds even for strongly interacting field theories, for example QCD. The general argument will be given in detail below, using Euclidean path integrals.

Exercise: Rindler Modes

Work through the details of section 9.5 in Carroll's GR textbook *Spacetime and Geometry*.

Exercise: Rindler Hamiltonian

Difficulty level: a few lines

To be more explicit let's write the Rindler Hamiltonian in terms of the stress tensor $T_{\mu\nu}$. Work in 2D for simplicity, but it easily generalizes. The original Minkowski Hamiltonian is

$$H = \int_{-\infty}^{\infty} dx\, u^\mu T_{\mu\nu} \zeta^\nu_{(t)} = \int_{-\infty}^{\infty} dx\, T_{tt} \ , \tag{3.26}$$

where the integral is over a fixed-t spatial slice, $u^\mu = \partial_t$ is the timelike normal to this slice, and $\zeta_{(t)} = \partial_t$ is the Killing vector associated to Minkowski-time translations. Similarly, the Rindler Hamiltonian is

$$H_\eta = \int_{-\infty}^{\infty} dx\, \tilde{u}^\mu T_{\mu\nu} \zeta^\nu_{(\eta)} = \int_{-\infty}^{\infty} dx\, \left(x^2 T_{tt}\right) \ . \tag{3.27}$$

where $\tilde{u}^\mu$ is the timelike normal to a fixed-η slice, and $\zeta_{(\eta)}$ is the Killing vector associated to η-translations (which are boosts in the original Minkowski coordinates).

Derive the second equality in (3.27).[19]

3.5 Hawking radiation

We have shown that Unruh observers see a heat bath set by the periodicity in imaginary time. We have also seen that the near-horizon region of a black hole is Rindler space. Putting these two facts together we get *Hawking radiation*: black holes radiate like a blackbody at temperature T. Some comments are in order:

1. This section has argued intuitively for Hawking radiation but don't be disturbed if you find the argument unconvincing. There are two ways to give a more

[19] I might have missed some numerical factors in that equation, if so then find them!

explicit and more convincing derivation. One is to match in modes to out modes and calculate Bogoliubov coefficients; I recommend you read this calculation in Carroll's book. The other method is using Euclidean path integrals to put the imaginary-time trick on a solid footing. We will follow this latter method in the next section.

2. This is the same T that appeared in the 1st law of thermodynamics. Historically, the 1st and 2nd law were discovered before Hawking radiation. Since T and S show up together in the 1st law, it was only possible to fix each of them up to a proportionality factor. This ambiguity was removed when Hawking discovered (to everyone's surprise) that black holes actually radiate.

3. In this derivation of Hawking radiation we have chosen a particular state by applying the imaginary-time periodicity trick. In the Unruh discussion, this trick selected the Minkowski vacuum. Another way to say it is that the imaginary-time trick picks a state in which the stress tensor is regular on the past and future Rindler horizon. Therefore, by applying this trick to black holes, we have selected a particular quantum state which is regular on the past and future event horizon. This state is called the *Hartle-Hawking vacuum.* Physically, it should be interpreted as a black hole in thermodynamic equilibrium with its surroundings.

   Others commonly discussed are:

   (a) The *Boulware* vacuum is a state with no radiation. In Rindler space, it corresponds to the Rindler vacuum, $\rho_{Rindler} = |0\rangle\langle 0|$. This state is singular on the past and future Rindler/event horizons, so is not usually physically relevant.

   (b) The *Unruh* vacuum is a state in which the black hole radiates at temperature T, but the surroundings have zero temperature. This is the state of an astrophysical black hole formed by gravitational collapse. It is regular on the future event horizon, but singular on the past even horizon (which is OK because black holes formed by collapse do not have a past event horizon). If we put reflecting boundary conditions far from the black hole to confine the radiation in a box, or if we work in asymptotically anti-de Sitter space, then the Unruh state eventually equilibrates so at late times is identical to the Hartle-Hawking state.

4. If you stand far from a black hole, you will actually not quite see a blackbody. A Rindler observer sees an exact blackbody spectrum, as does an observer hovering

near a black hole horizon. But far from the black hole, the spectrum is modified by a ‘greybody factor’ which accounts for absorption and re-emission of radiation by the intervening geometry. Far from a black hole, the occupation number of a mode with frequency ω is

$$\langle n_\omega \rangle = \frac{1}{e^{\beta\omega} - 1} \times \sigma_{abs}(\omega) \ . \tag{3.28}$$

The first term is the blackbody formula and the second term is the greybody factor. The greybody factor is equal to the absorption cross-section of a mode with frequency ω hitting the black hole, since transmission *into* the black hole is equal to transmission *out of* the black hole.

5. In Rindler space, an observer on a geodesic (*i.e.,* Minkowski observer) falls through the Rindler horizon, and this observer does not see the Unruh radiation. Similarly, you might expect that freely falling observers jumping into a black hole will not see Hawking radiation. This is almost correct, as long as the infalling observer is near the horizon in the approximately-Rindler region, but not entirely — the potential barrier between the horizon and $r = \infty$ causes some of the radiation to bounce back into the black hole, and this can be visible to an infalling observer.