25 Thermodynamics of 2d CFT

In this lecture we will discuss the thermodynamics of 2d CFT, including the torus partition function and the famous Cardy formula for CFT entropy. Later we'll come back to holography, but this note is entirely in CFT.

The partition function of a theory at finite temperature is

$$Z = \text{ Tr } e^{-\beta H} = \sum_{\text{states}} e^{-\beta E} \ . \tag{25.1}$$

This is a sum over states in some Hilbert space; that Hilbert space depends on our choice of the space in which the theory lives, *i.e.,* our choice of boundary conditions on the fields. If we choose space to be a circle, then

$$Z = \sum_{\text{states}} e^{-\beta E_{cyl}} = \sum_{\text{states}} e^{-\beta(\Delta - \frac{c}{12})} \ , \tag{25.2}$$

where Δ is the scaling dimension of the state. The second equality comes from (23.64), and assumes $L_{cyl} = 2\pi$.

As usual, the trace (25.1) is equal to a path integral in periodic imaginary time, with period β. Since space is also periodic, this is a path integral on a torus, with a 'space' identification and a 'thermal' identification:

$$w \sim w + L \sim w + i\beta \ . \tag{25.3}$$

$Z(\beta)$ is equal to the Euclidean path integral on the torus (25.3).

Our aim is to compute this path integral at high temperature. First we discuss general properties of CFT on a torus in the next couple subsections.

25.1 A first look at the S transformation

Consider a 2d QFT (not necessarily conformal yet) on a Euclidean torus. The most general torus is specified by two lattice vectors $\vec{v}_1, \vec{v}_2$ on the (t_E, ϕ) plane, meaning that we identify all points related by

$$(t_E, \phi) \sim (t_E, \phi) + m\vec{v}_1 + n\vec{v}_2, \qquad m, n \in \mathbf{Z} \ . \tag{25.4}$$

If the theory is rotationally invariant (*i.e.,* if its Lorentzian counterpart is Lorentz invariant), then we may w.l.o.g. rotate $\vec{v}_1$ to lie on the t_E axis. If the theory is scale invariant, then we can also w.l.o.g. set its length to $\vec{v}_2 = (0, 2\pi)$. Thus in a conformal field theory, we are led to consider the theory on the torus

$$(t_E, \phi) \sim (t_E + \beta, \phi + \theta) \sim (t_E, \phi + 2\pi) \ , \tag{25.5}$$

where β and θ are arbitrary real numbers.

In the complex coordinate $z = \frac{1}{2\pi}(\phi + it_E)$, the torus is specified by a complex number

$$\tau = \frac{1}{2\pi}(\beta + i\theta) \ . \tag{25.6}$$

The identifications are

$$z \sim z + 1 \sim z + \tau \ . \tag{25.7}$$

The path integral on this torus will be denoted $Z(\tau, \bar{\tau})$. τ is called the modulus, or complex structure modulus of the torus.

Converting to trace

The path integral on this torus can be converted to operator language by declaring that ϕ is 'space' and t_E is 'time.' Then we take states on a constant-t_E surface and evolve them along the vector $\beta\partial_{t_E} + i\theta\partial_\phi$. Thus the path integral can be rewritten as

$$Z(\tau, \bar{\tau}) = \text{ Tr }_{\mathcal{H}(0,2\pi)} e^{-\beta H + i\theta J} \tag{25.8}$$

where J is the angular momentum (which, by convention, generates motion along $-\partial_\phi$, hence the sign flip). In the subscript, we have noted explicitly what Hilbert space to trace over: it is the Hilbert space of states defined on the spatial slice where $\phi \in [0, 2\pi]$ with t_E held fixed. That is, the fields obey the boundary condition $X(t_E, \phi) = X(t_E, \phi + 2\pi)$. This is not standard notation; usually people just write ' Tr ', with $\mathcal{H}(0, 2\pi)$ implicit.

Large conformal transformations

We already know how conformal transformations act on the plane, as holomorphic/anti-holomorphic coordinate changes. Now we want to examine conformal symmetry on the torus. All of the usual conformal transformations $z \to z + \epsilon(z)$ are still symmetries, but there are also new, 'large' conformal transformations, which cannot be continuously connected to the identity.

The S transformation

One way to see this is to re-slice the torus path integral in a different way. We'll start with an intuitive explanation in the simplest case $\theta = 0$, then come back to the general story below. The torus is Euclidean, so we are free to switch the roles of t_E and ϕ when we construct the trace by declaring t_E is 'space' and ϕ is 'time'. Then following the usual logic, we find

$$Z(\tau, \bar{\tau}) = \text{ Tr }_{\mathcal{H}(\beta,0)} e^{-2\pi J} . \tag{25.9}$$

Therefore we have written the same path integral in two different ways (25.8) and (25.9),

$$Z(\tau, \bar{\tau}) = \text{ Tr }_{\mathcal{H}(0,2\pi)} e^{-\beta H} = \text{ Tr }_{\mathcal{H}(\beta,0)} e^{-2\pi J} . \tag{25.10}$$

This is true in any respectable quantum field theory. In a general (non-conformal) QFT, the Hilbert spaces in these two expressions are not the same. The first is the Hilbert space on the circle $\phi \sim \phi + 2\pi$, and the second is the Hilbert space on the circle $t_E \sim t_E + \beta$. Now, in a scale-invariant theory, these are related by a rotation by 90° followed by a rescaling by $2\pi/\beta$. Thus, in a CFT,

$$\text{Tr }_{\mathcal{H}(\beta,0)} e^{-2\pi J} = \text{ Tr }_{\mathcal{H}(0,2\pi)} e^{-\frac{4\pi^2}{\beta} H} . \tag{25.11}$$

Here the rotation by 90° sends $J \to H$, and the rescaling inserts the factor of $\frac{2\pi}{\beta}$.

It follows that

$$Z(\beta) = \text{ Tr } e^{-\beta H} = Z\left(\frac{4\pi^2}{\beta}\right) . \tag{25.12}$$

This is called the S transformation. It came from a large conformal transformation that swapped space and Euclidean time, and it relates the thermodynamics at high temperatures to the thermodynamics at low temperatures.

25.2 $SL(2,Z)$ transformations

The S transformation is one of an infinite group of large conformal transformations on the torus. These come from all the different ways of slicing the torus. We can think of these as ways of choosing the fundamental domain for the torus on the z-plane. The usual fundamental domain is the tilted rectangle with

$$z \sim z+1 \sim z+\tau \ . \tag{25.13}$$

But we can instead choose the even-more-tilted rectangle

$$w \sim w+1 \sim w+\tau+1 \ . \tag{25.14}$$

This is precisely the same torus, since the lattice $m+n\tau$ with $m,n \in \mathbf{Z}$ is unchanged. From the point of view of the z coordinate, the w coordinate system 'winds' around the torus:

$$FIGURE: Two tori, winding coords \tag{25.15}$$

This coordinate transformation is called 'T'. It acts on the modulus as

$$T \ : \quad \tau \to \tau+1 \ . \tag{25.16}$$

The winding means that this coordinate transformation cannot be continuously deformed to the identity, *i.e.,* there is no infinitessimal version.

Every theory, conformal or not, is invariant under T:

$$Z(\tau+1,\bar{\tau}+1) = Z(\tau,\bar{\tau}) \ . \tag{25.17}$$

This is because we haven't actually done anything, we've just rewritten the same torus path integral in a different coordinate system. We can also check explicitly from the trace formula:

$$Z(\tau+1,\bar{\tau}+1) = \text{ Tr } e^{-\beta H + i(\theta+2\pi)J} \ . \tag{25.18}$$

For a theory with only bosons, this is equal to $Z(\tau,\bar{\tau})$ since angular momentum is integer quantized. In a theory with fermions, we have to be more careful about imposing boundary conditions on the fermions, but in the end it still works.

There are in fact an infinite number of ways to slice the torus. The general choice of lattice vectors v_1', v_2' that generate the same lattice is

$$\begin{pmatrix} v_1' \\ v_2' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} , \tag{25.19}$$

where

$$a, b, c, d \in \mathbf{Z} \qquad \text{and} \qquad ad - bc = 1 . \tag{25.20}$$

The matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is therefore an element of the group $SL(2, Z)$— *i.e.,* , 2x2 matrices with integer elements and unit determinant. This re-slicing maps

$$\tau \to \frac{a\tau + b}{c\tau + d} . \tag{25.21}$$

$SL(2, Z)$ is generated by the S and T transformations. The S transformation is the matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, which acts as

$$S : \quad \tau \to -1/\tau . \tag{25.22}$$

Above, we discussed the S transformation with zero angular potential, $\tau = i\beta/(2\pi)$. In this case $S : \beta \to \frac{4\pi^2}{\beta}$ as claimed above.

Unlike T, the other $SL(2, Z)$ transformations are *not* symmetries of a general QFT. This is because the new space circle, which defines the Hilbert space, is not the same as the old space circle. It is only in a scale-invariant theory that we can rescale these two circles and see that they have the same Hilbert space. Thus in conformal field theory, $SL(2, Z)$ is a symmetry:

$$Z(\tau, \bar{\tau}) = Z\left(\frac{a\tau + b}{c\tau + d}, \frac{a\bar{\tau} + b}{c\bar{\tau} + d}\right) . \tag{25.23}$$

25.3 Thermodynamics at high temperature

(We will restrict to the case $\theta = 0$, but angular potential can be included at the expense of slightly more complicated formulas.) First let's calculate the partition function at very low temperature:

$$Z(\beta) = \sum_{\text{states}} e^{-\beta E} \approx e^{-\beta E_{vac}} \quad (\beta \to \infty) \ . \tag{25.24}$$

This is simply the statement that at very low temperature, the vacuum state dominates the canonical ensemble. Above we found the Casimir energy $E_{vac} = -\frac{c}{12}$, so

$$Z(\beta) \approx e^{\frac{\beta c}{12}} \quad (\beta \to \infty) \ . \tag{25.25}$$

Now, what about high temperatures? Modular invariance requires

$$Z(\beta) = Z(4\pi^2/\beta) \ , \tag{25.26}$$

so we can repeat the derivation at low temperature using the S-transformed partition function. Replacing $\beta \to 4\pi^2/\beta$ in (25.25) gives

$$Z(\beta) \approx e^{\frac{\pi^2 c}{3\beta}} \quad (\beta \to 0) \ . \tag{25.27}$$

The corresponding free energy, defined by $Z = e^{-\beta F}$, is

$$F = -\frac{\pi^2}{3} c T^2 \left(\frac{V}{2\pi} \right) \ , \tag{25.28}$$

where we've written the formula for a general 'volume', which we've been assuming is 2π. This is a remarkable formula. The scaling with the temperature in the free energy is fixed by dimensional analysis in a CFT, but the coefficient is not. The high-temperature free energy dominated by very heavy states, and is usually impossible to calculate in a strongly interacting QFT. But in this case, modular invariance relates it to the Casimir energy of the vacuum state.

There are two immediate consequences. First, this confirms once again that c should be interpreted as a measure of degrees of freedom. Second, it fixes the asymptotic density of states. To see this, write the result as

$$\sum_{states} e^{-\beta E} \approx \int dE \rho(E) e^{-\beta E} \approx e^{\pi^2 c/(3\beta)} \qquad (\beta \to 0) \ . \tag{25.29}$$

The first sum is over all states of the theory; the integral is over energies, with $\rho(E)$ the density of states.

The rhs is very singular as $\beta \to 0$. Each individual term on the lhs is regular at $\beta = 0$, so the singularity can only come from the infinite sum. The strength of the divergence must be related to the growth of $\rho(E)$ as $E \to \infty$.

To make this more quantitative, we can use standard thermodynamic formulae. The thermodynamic entropy and energy are

$$S = (1 - \beta\partial_\beta) \log Z = \frac{2\pi^2 c}{3\beta} \tag{25.30}$$

$$E(\beta) = -\partial_\beta \log Z = \frac{\pi^2 c}{3\beta^2} \ . \tag{25.31}$$

Putting these together, we have

$$S(E) = 2\pi\sqrt{\frac{c}{3}E} \ . \tag{25.32}$$

As usual in stat mech, this formula should give the density of states via

$$\rho(E) = e^{S(E)} \ . \tag{25.33}$$

As a check, let's plug this into the partition function and see that it reproduces the expected singularity as $\beta \to 0$:

$$Z(\beta) \approx \int dE \rho(E) e^{-\beta E} \tag{25.34}$$

$$\approx \int dE \exp\left(S(E) - \beta E\right) \tag{25.35}$$

$$\approx \exp\left(S(E_*) - \beta E_*\right) \tag{25.36}$$

where E_* is the saddlepoint value, defined by

$$S'(E_*) - \beta = 0 \ . \tag{25.37}$$

This saddlepoint is just the usual thermodynamic value of the energy. Using (25.32), finding the saddlepoint, and plugging in, we find

$$Z(\beta) \approx \exp\left(\frac{\pi^2 c}{3\beta}\right) \tag{25.38}$$

as claimed. (This is not a surprise; the whole point of thermodynamics and Legendre transforms is to solve this saddlepoint equation for you.)

The entropy formula (25.32) is often called the Cardy formula. It applies to any CFT as $E \to \infty$.

Exercise: Cardy formula with angular potential
Derive the asymptotic density of states $\rho(E, J)$ by applying modular invariance to the partition function including angular potential, $Z(\tau, \bar{\tau}) = \text{ Tr } e^{-\beta H + i\theta J}$.

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26 Black hole microstate counting

26.1 From the Cardy formula

The BTZ metric,

$$ds^2 = -\left(\frac{r^2}{\ell^2} - 8M\right) dt^2 + \frac{dr^2}{r^2/\ell^2 - 8M} + r^2 d\phi^2 \tag{26.1}$$

with $\phi \sim \phi + 2\pi$, has energy $E = M$ and entropy

$$\begin{aligned} S &= \frac{\text{area}}{4} \\ &= \frac{1}{4} 2\pi \rho_+ \\ &= \frac{\pi}{2}\sqrt{8M\ell} \\ &= 2\pi\sqrt{\frac{c}{3}E}\ . \end{aligned} \tag{26.2}$$

In the last line we used the Brown-Henneaux central charge, $c = 3\ell/2$ (with $G_N = 1$).

This exactly matches the Cardy formula, (25.32).* This match was found by Strominger in 1997. What is important about this result is that the Cardy formula *counts microstates in statistical mechanics*. They are microstates in the dual CFT, but by holographic duality, they must be microstates of quantum gravity in AdS_3 as well. We've just counted them without actually enumerating them, but a great deal of progress has been made enumerating them as well.

26.2 Strominger-Vafa

Historically, the first black hole microstate counting was a string theory calculation by Strominger and Vafa. It gave the same answer as (26.2). This calculation was important because it laid to rest any final doubts about whether black hole thermodynamics is *really* thermodynamics (*i.e.*, coming from stat mech) or just a mysterious analogy. In quantum gravity, black hole entropy counts microstates:

$$S_{BH}(E) \approx \log \rho_{QG}(E)\ , \tag{26.3}$$

where ρ_{QG} is the density of states in quantum gravity. This is an extraordinary window from low-energy physics into the theory of quantum gravity well above the Planck scale.

*Actually the Cardy formula was for $E \to \infty$, whereas the black hole formula is for $E > 0$. It is possible to get the match at all energies but requires further input from string theory, or some further assumptions about the spectrum of the CFT.