24 The stress tensor in 3d gravity

Now we will compare the stress tensor of 3d gravity to our results of the previous section. Consider the asymptotically-AdS_3 metric

$$ds^2 = \frac{\ell^2}{r^2}dr^2 + \frac{r^2}{\ell^2}dzd\bar{z} + h_{zz}dz^2 + h_{\bar{z}\bar{z}}d\bar{z}^2 + 2h_{z\bar{z}}dzd\bar{z} \ . \tag{24.1}$$

We have not written every possible term in the perturbation $h_{\mu\nu}$, but it turns out that other terms can be removed by a diff. Also, we will keep only the leading term in $h_{\mu\nu}$ at large r, *i.e.,* near the boundary, so we can assume that $h_{\mu\nu}$ is independent of r.

The Einstein equations imply that the perturbation is traceless and conserved:

$$h_{z\bar{z}} = 0 \tag{24.2}$$

$$\partial h_{\bar{z}\bar{z}} = \bar{\partial} h_{zz} = 0 \ . \tag{24.3}$$

24.1 Brown-York tensor

The stress tensor of this geometry was computed in an exercise from one of the early lectures. Let's briefly review how this works. The Brown-York stress tensor is*

$$T_{ij} \equiv -\frac{4\pi}{\sqrt{g}}\frac{\delta S^{on-shell}_{Einstein}}{\delta g^{ij}} \tag{24.4}$$

$$= -\frac{1}{4}\left(K_{ij} - Kg_{ij} - \frac{1}{\ell}g_{ij}\right) \ . \tag{24.5}$$

The first two terms came from varying the Einstein action plus the Gibbons-Hawking boundary term. The last term comes from the counterterm, with the coefficient set in order to make the answer finite as $r \to \infty$. Plugging the metric (24.1) into (24.5), using (24.2) and doing a lot of work, eventually

$$T_{zz} = -\frac{1}{4\ell}h_{zz} \ , \qquad T_{\bar{z}\bar{z}} = -\frac{1}{4\ell}h_{\bar{z}\bar{z}} \ . \tag{24.6}$$

*We've changed conventions by a factor of 2π compared to some earlier lectures. This is just a choice, made to agree with our convention for the CFT stress tensor.

Thus the Brown-York metric is traceless, conserved, and therefore holomorphic/anti-holomorphic just like in CFT.

24.2 Conformal transformations and the Brown-Henneaux central charge

Under diffeos, the metric (24.1) transforms as

$$\delta g_{\mu\nu} = \mathcal{L}_\zeta g_{\mu\nu} \ . \tag{24.7}$$

What vector fields ζ preserve the form of the metric (24.1)? The answer is

$$\begin{aligned} z &\to z + \epsilon(z) - \frac{\ell^4}{2r^2}\bar{\epsilon}''(\bar{z}) \\ \bar{z} &\to \bar{z} + \bar{\epsilon}(\bar{z}) - \frac{\ell^4}{2r^2}\epsilon''(z) \\ r &\to r - \frac{r}{2}\epsilon'(z) - \frac{r}{2}\bar{\epsilon}'(\bar{z}) \end{aligned} \tag{24.8}$$

for arbitrary functions $\epsilon(z)$ and $\bar{\epsilon}(\bar{z})$. Near the boundary, these act on $z, \bar{z}$ just like conformal transformations. The extra ∂_r piece acts as a rescaling.

Thus transformations of AdS_3 that preserve the asymptotics of the metric coincide with 2d conformal transformations!

Let's set $\bar{\epsilon} = 0$ and focus on the holomorphic conformal transformations. Under (24.8), the metric transforms as

$$ds^2 \to ds^2 + \left(-2h_{zz}\epsilon' - \epsilon h'_{zz} + \frac{\ell^2}{2}\epsilon'''\right) dz^2 \tag{24.9}$$

Thus the dz^2 piece of the metric, which we interpreted as the gravitational stress tensor up to a factor of $-1/4\ell$, transforms as

$$\delta_\epsilon T = -\epsilon \partial T - 2T\partial\epsilon - \frac{\ell}{8}\epsilon''' \ . \tag{24.10}$$

This is exactly the transformation law in 2d CFT derived in (23.53). Comparing the coefficient of the anomalous term, we see

$$c = \frac{3\ell}{2G_N} \tag{24.11}$$

where we've reinserted a factor of G_N (previously set to 1) by dimensional analysis.

This is called the *Brown-Henneaux central charge*, after Brown and Henneaux who computed it way back in 1987 – long before AdS/CFT, and even well before the discovery of the BTZ black hole or the Brown-York stress tensor. They used a different method, based on conserved charges, which directly produces the Virasoro algebra (23.56) as the asymptotic symmetry group. As far as I know, they did not recognize the relation to the 2d conformal group.

24.3 Casimir energy on the circle

Recall the metric of the 3d black hole (BTZ):

$$ds^2 = -(\frac{r^2}{\ell^2} - 8M)dt^2 + \frac{dr^2}{r^2/\ell^2 - 8M} + r^2 d\phi^2 \tag{24.12}$$

It is up to us whether to identity $\phi \sim \phi + 2\pi$ (since there is no conical defect even if we leave $\phi \in [-\infty, \infty]$). The BTZ black hole is the solution with $\phi \sim \phi + 2\pi$. The boundary of this spacetime is the Lorentzian (t, ϕ) cylinder, so this is dual to the CFT on a cylinder.

In an exercise in a previous lecture you computed the energy of this solution, and found $E = M$.

Global AdS can also be written in the form (24.12), by choosing $M = -\frac{\ell}{8}$. Therefore the gravitational energy of the groundstate on the cylinder is

$$E_{vac} = -\frac{\ell}{8} = -\frac{c}{12} \ , \tag{24.13}$$

where c take the Brown-Henneaux value (24.11).

This is equal to the Casimir energy of a 2d CFT (23.63). It was actually guaranteed to agree once we found the transformation law (24.10) agrees with CFT, because the finite version of this infinitessimal transformation must be the Schwarzian derivative, on the gravity side just as it was in CFT.