23 The Stress Tensor in 2d CFT

In the last few lectures, we will go into more depth on the AdS_3/CFT_2 correspondence. First we need to cover some more ground in the basics of 2d CFT.

References: For 2d CFT I recommend: Chapter 2 of Polchinski's text; Chapter 4 of Kiritsis's text; the big book of Di Francesco et al; and the string theory lectures notes by David Tong, available online. For the optimal introduction to the subject, I recommend working through the chapter of Tong's notes first, then working through chapters 4-6 of Di Francesco et al.

23.1 Infinitessimal coordinate changes

Recall that in two dimensions, with complex coordinates in Euclidean $\mathbf{R}^2$,

$$ds^2 = dz d\bar{z} \ , \tag{23.1}$$

conformal transformations are holomorphic coordinate changes:

$$w = w(z), \quad \bar{w} = \bar{w}(\bar{z}) \ . \tag{23.2}$$

The coordinate change

$$z' = z + \epsilon(z) \tag{23.3}$$

corresponds to the vector field

$$\zeta^\mu \partial_\mu = -\epsilon(z)\partial_z \tag{23.4}$$

They act on fields as

$$\phi'(z', \bar{z}') = \phi(z', \bar{z}') + \zeta^\mu \partial_\mu \phi \tag{23.5}$$

The infinitessimal generators can be taken as

$$\zeta_n = -z^{n+1}\partial_z, \quad \bar{\zeta}_n = -\bar{z}^{n+1}\partial_{\bar{z}} \ . \tag{23.6}$$

The conformal generators make an algebra

$$[\zeta_m, \zeta_n] = (m-n)\zeta_{m+n} \ , \tag{23.7}$$

and similarly for the barred generators. This is called the 'Witt algebra' (or centerless Virasoro algebra).

The algebra is infinite-dimensional. There is one subalgebra, which consists of the 3 generators $\zeta_{0,1,-1}$. These make the *global subalgebra* SL(2):

$$\zeta_{-1}, \zeta_0, \zeta_1 \quad \text{and} \quad \bar{\zeta}_{-1}, \bar{\zeta}_0, \bar{\zeta}_1 \quad \Rightarrow \quad SL(2,R) \times SL(2,R) \sim SO(2,2) \ . \tag{23.8}$$

It is called the global subalgebra because these are the only $\zeta_n's$ that are non-singular on the Riemann sphere. To see this, first look near $z \sim 0$. Clearly the vector field $\zeta_n = -z^{n+1}\partial_z$ is regular only for $n \geq -1$. Now, do the coordinate change $w = 1/z$, and look near $w \sim 0$. The vector field becomes

$$\zeta_n = -z^{n+1}\partial_z = -w^{-n-1}(-w^2)\partial_w \tag{23.9}$$

which is regular only for $n \leq 1$. So the generators in (23.8) are the only ones regular at both poles of the Riemann sphere.

23.2 The Stress Tensor

Translation invariance implies the action S is invariant under $x^\mu \to x^\mu + \epsilon^\mu$, for constant ϵ^μ. The classical stress tensor is defined by applying the Noether to this symmetry. Promoting ϵ^μ to an arbitrary function of x^μ and varying the action must give something proportional to $\partial\epsilon$,

$$\delta S = -2\int d^2z\sqrt{g}T^{\mu\nu}\partial_\mu\epsilon_n \ . \tag{23.10}$$

This defines the stress tensor*

$$T_{\mu\nu} = -\frac{4\pi}{\sqrt{g}}\frac{\delta S}{\delta g^{\mu\nu}} \ . \tag{23.11}$$

*Note the extra -2π compared to our when we computed gravitational stress tensors a while back. This is just a convention but will be important to remember when we compare the two.

It's conserved

The Noether procedure guarantees that

$$\nabla_\mu T^{\mu\nu} = 0 \ . \tag{23.12}$$

In a flat background (23.1), in complex coordinates the two components of this equation are

$$\bar{\partial} T_{zz} = 0 \ , \qquad \partial T_{\bar{z}\bar{z}} = 0 \ . \tag{23.13}$$

where we have introduced the shorthand

$$\partial \equiv \partial_z, \qquad \bar{\partial} = \partial_{\bar{z}} \ . \tag{23.14}$$

From (23.13), T_{zz} is a holomorphic function of z, and $T_{\bar{z}\bar{z}}$ is anti-holomorphic. These will be denoted

$$T(z) \equiv T_{zz}, \qquad \bar{T}(\bar{z}) = T_{\bar{z}\bar{z}} \ . \tag{23.15}$$

And traceless

So far we have used only translation invariance. In a theory that is classically scale invariant, we can also conclude that the trace of the stress tensor vanishes. The symmetry in this case is the infinitessimal rescaling

$$x^\mu \to x^\mu + \lambda x^\mu \ . \tag{23.16}$$

Since under a rescaling

$$\partial_\mu \epsilon_\nu = \lambda g_{\mu\nu} \ , \tag{23.17}$$

the variation of the action is

$$\delta S \propto \int d^2 z \sqrt{g} \lambda T^\mu_\mu . \tag{23.18}$$

Scale invariance implies that this integral vanishes; conformal invariance, or local scale invariance, means we can make $\lambda \to \lambda(z, \bar{z})$ so $T^\mu_\mu = 0$. In complex coordinates,

$$T_{z\bar{z}} = 0 \ . \tag{23.19}$$

This is the classical stress tensor. Even if it is traceless, the quantum stress tensor might have a non-zero trace, for two different reasons. First, the UV regulator introduces a scale, and may introduce a trace. In fact, in a renormalizable theory,

$$T^{\mu}_{\mu}(x) = \sum_i \beta_{g_i} O_i(x) \ , \tag{23.20}$$

where O_i are the relevant operators of the theory, g_i are the corresponding couplings, and β_g are their beta functions. So, for example, in massless QCD, although the classical stress tensor is traceless, the quantum stress tensor has a contribution from the non-zero QCD beta function. In a CFT, all the beta functions are exactly zero, so the equation

$$T^{\mu}_{\mu} = 0 \tag{23.21}$$

is true as an operator statement. The phrase "as an operator statement" means the equation is true *inside* correlation functions, up to delta-functions where T^{μ}_{μ} hits other operator insertions (we'll see some of these delta functions below).

The second origin of a non-zero trace is a quantum anomaly. This happens even in CFT, if we place the theory on a curved background. This is called the Weyl anomaly and is important but we probably won't have time to cover it.

Noether currents for conformal symmetries

$T(z)$ is the Noether current for translations. That is, the current J^{μ} with

$$J^{\bar{z}} = T(z) \tag{23.22}$$

is the current associated to $z \to z+$const. What are the Noether currents associated to the general conformal transformation $z \to z + \epsilon(z)$? These are simply

$$J^{\bar{z}} = \epsilon(z) T(z) \ . \tag{23.23}$$

This is sort of obvious; to reproduce it from the Noether procedure, you can promote $\epsilon(z) \to \epsilon(z, \bar{z})$ and apply the usual Noether procedure.

As expected, the current is conserved,

$$\partial_\mu J^\mu = \bar{\partial}(\epsilon T) = 0 \ . \tag{23.24}$$

23.3 Ward identities

Ward identities are the quantum version of the Noether procedure.

Suppose we have a general symmetry $\phi' = \phi + \epsilon\delta\phi$. The fact that this is a symmetry means the action and the path integral measure are invariant,

$$S[\phi'] = S[\phi], \qquad D\phi' = D\phi \ . \tag{23.25}$$

Thus, promoting $\epsilon \to \epsilon(x^\mu)$,

$$\int D\phi e^{-S[\phi]} = \int D\phi' e^{-S[\phi']} \tag{23.26}$$

$$= \int D\phi e^{-S[\phi] - \int J^\mu \partial_\mu \epsilon} \tag{23.27}$$

$$= \int D\phi (1 - \int J^\mu \partial_\mu \epsilon) e^{-S[\phi]} \tag{23.28}$$

The first line is just renaming a dummy variable; the second line defines the current, J^μ, which may have contributions from both the classical action and the measure; and the third line expands to linear order. It follows that

$$\langle \int J^\mu \partial_\mu \epsilon \rangle = 0 \tag{23.29}$$

for all ϵ, and so

$$\langle \partial_\mu J^\mu \rangle = 0 \ . \tag{23.30}$$

This is the quantum version of the Noether procedure. The same exact steps, starting instead with $\int D\phi O_1(x_1) \cdots O_n(x_n) e^{-S[\phi]}$ can be used to show that

$$\langle \partial_\mu J^\mu(y) O_1(x_1) \cdots O_n(x_n) \rangle = 0 \quad \text{if} \quad x_i \neq y \ . \tag{23.31}$$

The restriction to $x_i \neq y$ is necessary for the derivation to work. Just set the support of

ϵ to a small circle around y that does not include any of the other operator insertions, and repeat the steps above.

The equation (23.31) means

$$\partial_\mu J^\mu = 0 \tag{23.32}$$

This is an operator equation, ie it holds inside correlators, up to delta functions.

If there are insertions that collide with $\partial_\mu J^\mu$, we have to be more careful. Suppose $x_1 = y$. Then when we do the transformation inside the path integral, the transformation also affects this operator insertion,

$$O_1 \to O_1 + \epsilon\delta O_1\ . \tag{23.33}$$

So now,

$$\int D\phi O_1(x_1)\cdots O_n(x_n)e^{-S[\phi]} = \int D\phi(1 - \int J^\mu\partial_\mu\epsilon)(O_1 + \epsilon\delta O_1)O_2\cdots O_n e^{-S[\phi]} \tag{23.34}$$

and the conservation law is modified to (restoring some dropped constants)

$$-\frac{1}{2\pi}\int_{D(y)}\partial_\mu\langle J^\mu(y)O_1(x_1)\cdots O_n(x_n)\rangle = \langle\delta O_1(x_1)O_2(x_2)\cdots O_n(x_n)\rangle, \tag{23.35}$$

where $D(y)$ is a disk enclosing y, where we've chosen ϵ to be non-zero. Allowing for any of the operators to collide with the current, (23.35) becomes

$$\partial_\mu\langle J^\mu(y)O_1(x_1)\cdots O_n(x_n)\rangle = \sum_i \delta(y - x_i)\langle O_1\cdots\delta O_i\cdots O_n\rangle\ . \tag{23.36}$$

(23.36) is called the Ward identity.

As a residue

In two dimensions, using complex coordinates we can write (23.36) in a nice way. By Stokes, we have

$$\int_{D(y)}\partial_\mu J^\mu \sim \oint_y (J_z dz - J_{\bar z} d\bar z) \tag{23.37}$$

and

$$\frac{i}{2\pi}\oint_x dz J(z)O(x) = -\text{res}_{z\sim x} J(z)O(x) \ . \tag{23.38}$$

Therefore, the Ward identity (23.36) can be written as the operator equation

$$\delta O(x) = -\text{res}_{z\sim x}\left[J(z)O(x)\right] \ . \tag{23.39}$$

Conformal Ward Identities

The Noether current for the conformal symmetry $z \to z + \epsilon(z)$, from (23.23), is

$$J(z) = \epsilon(z)T(z) \ . \tag{23.40}$$

Therefore the Ward identity for conformal transformations, allowing for both holomorphic and anti-holomorphic transformations, is

$$\delta_{\epsilon,\bar{\epsilon}} O(x) = -\text{res}_{z\sim x}\left[\epsilon(z)T(z)O(x)\right] \ . \tag{23.41}$$

23.4 Operator product expansion

The product of two local operators can be Taylor-expanded as local operators at a single point:

$$O_1(x)O_2(y) = \sum_j f_{12j}(x-y)O_j(u) \ . \tag{23.42}$$

The sum is over all local operators in the theory. This is called the operator product expansion (OPE). The OPE coefficients $f_{12j}(x-y)$ depend on the theory, but they are restricted by conformal invariance, so that in fact the f_{ijk}'s of primary operators determine the f_{ijk}'s of descendants as well.

The Ward identity, in the form (23.41), means that the singular terms in the $O(x)T(y)$ OPE are related to the conformal transformations of O.

Primaries

Recall that primary operators transform as

$$O'(z',\bar{z}') = \left(\frac{dz'}{dz}\right)^{-h}\left(\frac{d\bar{z}'}{d\bar{z}}\right)^{-\bar{h}} O(z,\bar{z}) \tag{23.43}$$

where $(h,\bar{h})$ are called the conformal weights of O. The corresponding infinitessimal transformation is

$$\begin{aligned}\delta_{\epsilon,\bar{\epsilon}}O(z,\bar{z}) &\equiv O'(z,\bar{z}) - O(z,\bar{z}) &(23.44)\\ &= -(hO\partial\epsilon + \epsilon\partial O) - (\bar{h}O\bar{\partial}\bar{\epsilon} + \bar{\epsilon}\bar{\partial}O) &(23.45)\end{aligned}$$

Comparing to (23.41), this means

$$T(z)O(w,\bar{w}) \sim \frac{hO(w,\bar{w})}{(z-w)^2} + \frac{\partial O(w,\bar{w})}{z-w}\ . \tag{23.46}$$

The symbol '$\sim$' means that we only written the singular terms in the OPE; there is also an infinite series of contributions regular at $z = w$. To check this gives the correct residue, expand

$$\epsilon(z)T(z)O(w) = T(z)\epsilon(w)O(w) + (z-w)T(z)\epsilon'(w)O(w) + \cdots \tag{23.47}$$

and look at the simple pole in (23.46).

Upshot

The lesson is that *singular terms in the $T\phi$ OPE contain exactly the same information as the transformations of ϕ under conformal symmetry.* The conformal algebra is one and the same as the data in the singular part of the OPE.

23.5 The Central Charge

Now we will examine the transformation of the stress tensor under conformal symmetry, or equivalently, the TT OPE. Much of the discussion in this section is easiest to understand first using an example like a free scalar field. This example is worked in every reference so I will not repeat here but encourage the reader to jump to the free scalar section of Polchinski, Kiritsis, or Tong's notes before continuing.

The stress tensor is not a primary, so we cannot plug $O = T$ into (23.46). But, it does behave similar to a primary under rescalings, with $(h, \bar{h}) = (2, 0)$:

$$T'(\lambda z) = \lambda^{-2} T(z) \ . \tag{23.48}$$

This is basically dimensional analysis; the energy should have mass dimension 1, and $E \sim \int T$. Similarly, $\overline{T}(\bar{z})$ has scaling weights $(h, \bar{h}) = (0, 2)$. Requiring both sides of the OPE to have the same scaling weight, it must have the form

$$T(z)T(w) = \frac{c/2}{(z-w)^4} + \frac{X_1(w)}{(z-w)^3} + \frac{X_2(w)}{(z-w)^2} + \frac{X_3(w)}{z-w} + \cdots \ , \tag{23.49}$$

where c is a number (the 2 is inserted by convention), X_1 is some field of dimension $(1, 0)$, X_2 is dimension $(2, 0)$, and X_1 is dimension $(3, 0)$. There are no terms more singular than $(z - w)^4$, since fields must have positive scaling weight in a unitary theory.* The exchange symmetry

$$T(z)T(w) = T(w)T(z) \tag{23.50}$$

set $X_1 = 0$. The OPE is always invariant under permutations (in Euclidean signature). One way to see this is that equal-time commutators in Lorentzian signature must vanish by causality, and these equal-time commutators correspond to permutations of the Euclidean correlator. This property is sometimes called locality or causality.

To fix the other two terms, we use the Ward identity for the scale symmetry (23.48). The Noether current for scale symmetry is, from (23.23)

$$J_{scale}(z) = zT(z) \ . \tag{23.51}$$

Then using the Ward identity (23.41) gives our final answer for the singular terms in the OPE:

$$T(z)T(w) \sim \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} \ . \tag{23.52}$$

*I will not explain why, but this can be found in the references, in the section on the state-operator correspondence and conformal reps. This also explains why c must be a number, not a field.

The constant c is called the central charge. For a free boson, it turns out to be $c = 1$, and for a free fermion $c = 1/2$. (see references).

Transformation law for the stress tensor

Knowing the TT OPE, we can use the Ward identity $\delta T(w) = -\text{res}_{z\sim w}[\epsilon(z)T(z)T(w)]$ to find how T transforms under a conformal symmetry:

$$\delta T = -\epsilon T' - 2\epsilon' T - \frac{c}{12}\epsilon''' \ . \tag{23.53}$$

The first two terms are the usual transformations of a primary. The last term is an 'anomolous' term coming from the central charge.

The finite transformation law, obtained by integrating (23.53) is

$$T'(z') = \left(\frac{dz'}{dz}\right)^{-2}\left[T(z) - \frac{c}{12}\{z', z\}\right] \ , \tag{23.54}$$

where

$$\{f(z), z\} \equiv \frac{f'''}{f'} - \frac{3}{2}\frac{(f'')^2}{(f')^2} \tag{23.55}$$

is called the *Schwarzian derivative*. The easiest way to derive (23.54) is to check that it agrees infinitessimally with (23.53), and that it composes correctly under multiple transformations.

Aside: Virasoro algebra

We won't cover the operator formalism in this course, so I will not discuss the Virasoro algebra in detail. Roughly, you can decompose the stress tensor into modes, $L_n \sim \oint dz z^{n+1}T(z)$. Then the TT OPE (23.52), combined with the Ward identity, become the Virasoro algebra

$$[L_m, L_n] = (m-n)[L_m - L_n] + \frac{c}{12}(m^3 - m)\delta_{m,-n} \ . \tag{23.56}$$

This is where the name 'central charge' comes from. It is another way of writing (23.52): the TT OPE, the Schwarzian derivative, and the Virasoro algebra all contain the same information.

23.6 Casimir Energy on the Circle

c is related to the Casimir energy of the theory on a circle. The mapping from the plane to the cylinder of radius L is

$$z = e^{2\pi w/L} . \tag{23.57}$$

The cylinder coordinate is identified $w \sim w + iL$, since this takes us around a circle on the z plane. Using the finite transformation law (23.54) gives

$$T_{cyl}(w) = \left(\frac{2\pi}{L}\right)^2 z^2 \left(T_{plane}(z) - \frac{c}{24z^2}\right) , \tag{23.58}$$

where now we're using 'plane' and 'cyl' to distinguish the stress tensor before and after the transformation.

Let's calculate the expectation value. On the plane, scale invariance sets all 1-point functions to zero:

$$\langle T_{plane}(z)\rangle = 0 . \tag{23.59}$$

This is because the only scale-invariant function of z with weight 2 is $1/z^2$, but this would not be translation invariant. Now using (23.58),

$$\langle T_{cyl}(w)\rangle = -\frac{c}{24}\left(\frac{2\pi}{L}\right)^2 . \tag{23.60}$$

This is a Casimir energy, in the usual sense: we started with a theory on a line (*i.e.,* space is a line), then imposed periodic boundary conditions with period L, and found energy $\sim 1/L$. The size of the Casimir energy is fixed by the central charge. This is our first indication that c might be a good way to the measure the degrees of freedom of a CFT.

To compute the value of the energy explicitly, choose real coordinates $w = \tau + i\phi$, where $\phi \sim \phi + L$. The energy is defined in the usual way by integrating the stress tensor over a fixed-time slice:

$$E_{cyl} = \frac{1}{2\pi} \in_0^L dx \langle T_{\tau\tau}^{cyl}(\tau = 0, x) \rangle \tag{23.61}$$

$$= \frac{1}{2\pi} \int_0^L dx \langle T_{ww}^{cyl} + T_{\bar{w}\bar{w}}^{cyl} \rangle \tag{23.62}$$

In the second line, we just did the change of coordinates $w = \tau + i\phi, \bar{w} = \tau - i\phi$, and used tracelessness of the stress tensor, $T_{w\bar{w}} = 0$. Evaluating this in vacuum gives the Casimir energy

$$E_{cyl}^{vac} = -\frac{c}{12}\frac{2\pi}{L} \ . \tag{23.63}$$

In a general state, (23.61) gives

$$E_{cyl} = \Delta - \frac{c}{12} \tag{23.64}$$

where

$$\Delta = \langle \frac{1}{2\pi i} \left(\oint dz z T(z) + \oint d\bar{z} \bar{z} \bar{T}(\bar{z}) \right) \rangle \ . \tag{23.65}$$

Recall that $J = zT(z)$ is the Noether current for scale transformations. Therefore, the rhs is the expectation value of the scale operator. In an eigenstate, Δ is the scaling dimension of that state. So (23.64) says that the energy on the cylinder is the scaling dimension on the plane, shifted by the central term. This makes sense, since time translations on the cylinder correspond to scale transformations on the plane.

Exercise: free boson

The action of a free boson is $S = \int d^2z \partial\phi\bar{\partial}\phi$. Use the Noether procedure to find the stress tensor. Then, compute the TT OPE and confirm that $c = 1$. (See Polchinski, Kiritsis, Di Francesco, or Tong's notes if you get stuck).