14 The Statement of AdS/CFT

14.1 The Dictionary

Choose coordinates

$$ds^2 = \frac{\ell^2}{z^2}(dz^2 + dx^2) \tag{14.1}$$

on Euclidean AdS_{d+1}, where x is a coordinate on R^d. The boundary is at $z = 0$.

We showed above that scattering problems in gravity map to correlation functions in CFT. In this relation the boundary value of the bulk field acted as a source for a CFT operator. This is generalized by the following statement of the AdS/CFT correspondence:

$$Z_{grav}[\phi_0^i(x); \partial M] = \left\langle \exp\left(-\sum_i \int d^d x \phi_0^i(x) O^i(x) \right) \right\rangle_{CFT\ on\ \partial M} \tag{14.2}$$

This is called the GKPW dictionary.[65] The index i runs over all the light fields in the bulk effective field theory, and correspondingly over all the low-dimension local operators in CFT.

The left-hand side

The lhs of (14.2) is the gravitational partition function in asymptotically AdS space. It is formally computed by the same path integral that we discussed in the context of black hole thermodynamics. Since AdS has a boundary, we must provide boundary conditions to define this path integral. The boundary conditions on bulk scalars are

$$\phi^i(z, x) = z^{d-\Delta}\phi_0^i(x) + \text{subleading as } \ z \to 0\ . \tag{14.3}$$

where the mass of the bulk scalar is related to the scaling dimension of the CFT operator by

$$m^2 = \Delta(d - \Delta)\ , \qquad \Delta = \frac{d}{2} + \sqrt{\frac{d^2}{4} + m^2\ell^2}\ . \tag{14.4}$$

[65] After hep-th/9802109 by Gubser, Klebanov and Polyakov and hep-th/9802150 by Witten. I highly recommend reading Witten's paper.

We will see below that (14.3) is the leading solution of the wave equation for a bulk scalar of mass m.

Similar statements apply to all bulk fields, including the metric, though the boundary condition and formula for the dimension is slightly modified for fields with spin. The boundary conditions on the metric involve a choice of topology as well as the actual metric, which is why we've indicated explicitly that Z_{grav} depends on the boundary manifold ∂M.

The right-hand side

The rhs of (14.2) is the generating functional of correlators in a CFT. In this equation the $\phi_0^i(x)$ are sources, and the $O^i(x)$ are CFT operators. Denoting the rhs of (14.2) by $Z_{cft}[\phi_0]$, correlation functions are computed in the usual way,

$$\langle O_1(x_1)\cdots O_n(x_n)\rangle_{CFT} \sim \frac{\delta^n}{\delta\phi_0^1(x_1)\cdots\delta\phi_0^n(x_n)} Z_{cft}[\phi_0]\Big|_{\phi_0^i=0} \ . \qquad (14.5)$$

The mapping

Each light field in gravity corresponds to a local operator in CFT. The spin of the bulk field is equal to the spin of the CFT operator; the mass of the bulk field fixes the scaling dimension of the CFT operator. Here are some examples:

Scalar: A bulk scalar field $\chi(z,x)$ is dual to a scalar operator in CFT. The boundary value of χ acts as a source in CFT. This is exactly the relationship we used in our derivation of the absorption cross section of the black string.

Graviton: Every theory of gravity has a massless spin-2 particle, the graviton $g_{\mu\nu}$. This is dual the stress tensor $T_{\mu\nu}$ in CFT. This makes sense since every CFT has a stress tensor. The fact that the graviton is massless corresponds to the fact that the CFT stress tensor is conserved. It also fixes the scaling dimension to $\Delta_T = d$. We will see this in more detail later.

Vector: If our theory of gravity has a spin-1 vector field A_μ, then the dual CFT has a spin-1 operator J_μ. If A_μ is massless, then $\Delta_J = d-1$ and J_μ is a conserved current. Otherwise, $\Delta_J > d-1$ and the current is not conserved.

This illustrated a general and important feature of AdS/CFT: *gauge symmetries in the bulk correspond to global symmetries in the CFT.*

This is UV complete.

Note that CFTs are UV complete. Therefore (14.2) is a non-perturbative formulation of a UV complete theory of quantum gravity. Shockingly, it is a definition of gravity from a QFT without gravity. This is very powerful because we understand QFT relatively well.

14.2 Example: IIB Strings and $\mathcal{N} = 4$ Super-Yang-Mills

In some sense, it is believed that the AdS/CFT correspondence as summarized by (14.2) holds for any theory of gravity and and CFT. That is, given a theory of gravity we can use it to *define* a CFT via (14.2), and (perhaps) vice-versa. But aside from certain examples, the correspondence is well defined and useful only in certain limits. To illustrate this we turn to a specific example where AdS/CFT is understood in great detail. This is the duality between IIB string theory and supersymmetric gauge theory:

$$\text{IIB strings on } \ AdS_5 \times S^5 \quad = \quad \text{Yang-Mills in 4d with } \ \mathcal{N} = 4 \text{ supersymmetry} \ .$$

The gravity side

The string theory has adjustable scales $\ell \equiv \ell_{AdS}$, the Planck scale ℓ_P, and the string scale ℓ_s. We do not need to use any details of string theory except to say that at low energies, the effective action is Einstein + Matter + higher curvature corrections suppressed by the string scale:[66]

$$S_{IIB} \sim \frac{1}{G_N} \int \sqrt{g} \left(R + L_{matter} + \ell_s^4 R^4 + \cdots \right) \tag{14.6}$$

The stringy states have masses of order $1/\ell_s^2$, so at energies below $1/\ell_s^2$ it is just an ordinary effective field theory like we discussed at the beginning of the course.

[66] There are no R^2 corrections allowed with this amount of supersymmetry, but there are similar examples with non-zero $\ell_s^2 R^2$ terms.

The CFT side

$\mathcal{N} = 4$ Super-Yang-Mills is a highly supersymmetric gauge theory in 4d. Its matter context is fixed uniquely by supersymmetry. It is just an $SU(N)$ gauge field plus all the matter fields required by supersymmetry, which include matrix-valued scalar fields transforming the adjoint representation of $SU(N)$ (unlike the fundamental representations we usually encounter in, say, QCD).

The gauge theory has two dimensionless parameters, N (ie the size of $SU(N)$) and the Yang-Mills coupling constant g_{YM}. Define the combination

$$\lambda = g_{YM}^2 N \ . \tag{14.7}$$

This is called the 't Hooft coupling. It turns out that gauge theory at large N is most naturally organized as an expansion in λ and $1/N$, rather than g_{YM} and $1/N$. This is roughly because there are N fields running in loops, which changes the expansion parameter from g_{YM}^2 to λ.

The mapping

The mapping from string theory parameters to CFT parameters is

$$\lambda \sim \left(\frac{\ell_{AdS}}{\ell_{string}}\right)^4 \tag{14.8}$$

and

$$\frac{\ell_{AdS}^{d-1}}{G_N} \sim \left(\frac{\ell_{AdS}}{\ell_P}\right)^{d-1} \sim N^2 \ . \tag{14.9}$$

(with known coefficients). We will see where this particular scaling comes from below in more generality. For now we just want to note that this is a *strong/weak duality*: when one side is easy, the other is (usually) hard. For example to have semiclassical Einstein gravity, both loops and higher curvature corrections must be suppressed on the gravity side. This means $N \gg 1$ and $\lambda \gg 1$ so the CFT is very strongly coupled. On the other hand if we consider a weakly coupled CFT, then $\ell_s \gg \ell_{AdS}$ so stringy/higher curvature corrections are not suppressed on the gravity side and this presumably behaves nothing like ordinary gravity. (This is related to so-called 'higher spin gravity' or 'Vasiliev gravity'.)

14.3 General requirements

Returning to AdS/CFT in general, we can make some similar observations about when it produces a nice semiclassical theory of gravity. This requires as least two things:

1. **Strongly coupled CFT.** If the CFT is weakly coupled, then there are too many operators. For example, a free scalar field ψ leads to conserved currents of every integer spin:[67]

$$\psi\partial_\mu\psi, \qquad \psi\partial_\mu\partial_\nu\psi, \qquad \psi\partial_\mu\partial_\nu\partial_\rho\psi, \quad \text{etc.} \tag{14.10}$$

   On the gravity side, this would require lots of massless or very light high-spin states. This is something we expect in string theory at high enough energies but not in our low energy effective field theory.

   So we must require that the CFT has a sparse spectrum of low-dimension operators. This is sometimes called a large 'gap' in the spectrum, meaning a gap between the low-energy fields and the stringy stuff. This can only happen at strong coupling, although there can also be strongly coupled theories with no gap which therefore do not have nice gravity duals.

2. **Large N_{dof}.** In the super-Yang-Mills example, we said $G_N \sim 1/N^2$ so that the large number of degrees of freedom is required for gravity to be weakly coupled. This is true in general, too. There are two ways to see this, both of which we will discuss in more detail later. I will purposely be a little vague about the definition of N_{dof} since there are several reasonable ways to define it, and they are all different.

   First, note that black hole entropy is $S \propto 1/G_N$, which is very large. Since entropy is the log of the density of states, this means holographic CFTs must have an enormous degeneracy of states at high energy. This means there are lots of degrees of freedom. For example, a 2d CFT consisting of N_b free bosons has $S(E) \propto \sqrt{N_b E}$.

   Second, we can roughly measure the degrees of freedom by looking at the stress-tensor 2pt function. This is fixed by conformal invariance up to a single coeffi-

[67] This is schematic, you must add corrections to these operators for them to be conserved by the EOM.

cient:

$$\langle T_{\mu\nu}(x) T_{\alpha\beta}(y) \rangle = c \times (\text{known function of } x, y) \ . \tag{14.11}$$

The coefficient c is a measure of degrees of freedom.[68] Consider again lots of free fields: the stress tensors add, so the total stress tensor will have a very big 2pt function.

On the gravity side, the stress tensor is dual to the graviton. We will see in detail below how to calculate corelators, but for now suffice it to say that $\langle TT \rangle_{cft}$ will be related to a graviton scattering experiment $\langle gg \rangle_{gravity} \sim 1/G_N$. Thus $c \sim 1/G_N$ and we see again that weakly coupled gravity requires an enormous number of degrees of freedom.

Clear there is a tension between requirements (1) and (2). We want lots of degrees of freedom, and lots of states at high energies, but very few states at low energies. Roughly speaking, you can think of this as the requirement that the CFT by *confining*: it has lots of states at high energies, but very few at low energies where quarks are confined. Later we will see a very direct link between black hole thermodynamics and confinement.

14.4 The Holographic Principle

Many years ago Bekenstein conjectured that the maximum entropy you can fit into a region of space is equal to the entropy of the corresponding black hole:

$$S_{max} = \frac{\text{area}}{4G_N} \ . \tag{14.12}$$

This is called the *Bekenstein bound*. The argument is simple. If you have lots of stuff in a region and $S_{stuff} > S_{blackhole}$, then you can throw in some more stuff and form a black hole. In doing so, the entropy of the system decreases! Therefore the second law requires a bound like (14.12).[69]

This bound inspired 't Hooft (in '93) and later Susskind (in '94) to argue that a theory

[68] But not an entirely satisfactory one. For example, it can increase under RG flow.

[69] In the last few years this bound has been understood much better using entanglement entropy. See for example 1404.5635 and references therein.

of quantum gravity must secretly live in fewer dimensions than our observed spacetime. This principle is realized concretely by AdS/CFT.